Hitting Matrix and Domino Tiling with Diagonal Impurities :

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Abstract

As a continuation to our previous work [9, 10], we consider the domino tiling problem with impurities. (1) if we have more than two impurities on the boundary, we can compute the number of corresponding perfect matchings by using the hitting matrix method[4]. (2) we have an alternative proof of the main result in [9] and result in (1) above using the formula by Kenyon-Wilson [6, 7] of counting the number of groves on the circular planar graph. (3) we study the behavior of the probability of finding the impurity at a given site when the size of the graph tends to infinity, as well as the scaling limit of those.

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1 Introduction

1.1 Background

Let $G = (V(G), E(G))$ be a graph. A subset M of $E(G)$ is called a **perfect matching**(or a **dimer covering**) on G if and only if for any $x \in V(G)$

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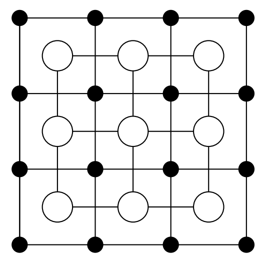

Figure 1.1: G_2 and G_1

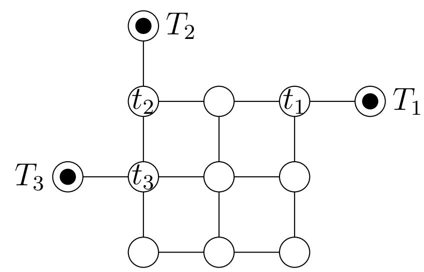

Figure 1.2: An example of $G_{1,T}$ for $k = 2$.

there exists $e \in M$ uniquely with $x \in e$. Let $\mathcal{M}(G)$ be the set of perfect matchings on G. The perfect matching problem was first studied by Kasteleyn, Temperley-Fisher, in the context of the statistical mechanics, and many papers appeared since then. In particular, for the perfect matching on the bipartite graphs, such as domino or lozenge tilings, many results have been known (e.g., [5] and references therein).

In this paper we consider the perfect matching on a non-biparite graph $G^{(k)}$ defined below. We use the same notation in [10]. Let G_2 be a subgraph of the square lattice, and let G_1 be its dual graph(Figure 1.1).

Let $G_{1,T}$ be the graph made by adding $(2k-1)$ vertices $T_1, T_2, \cdots, T_{2k-1}$, which we call the **terminal**, to the boundary ∂G_1 of G_1 (Figure 1.2).

We next superimpose G_2 and $G_{1,T}$, and we make a vertex wherever two edges cross, and call it a **middle vertex**. For the vertices in the boundary of G_1, we add extra edges toward the outer face of G_2, and make vertices, which we call the **boundary vertex**, wherever two edges meet(Figure 1.3).

By adding furthermore diagonal edges alternately, we have the graph

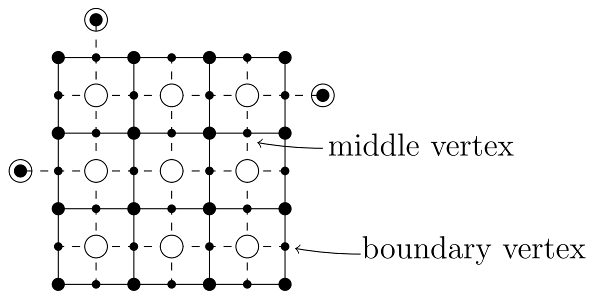

Figure 1.3: We superimpose G_2 and $G_{1,T}$ and then add middle and boundary vertices

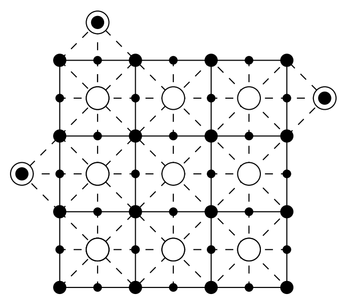

Figure 1.4: Graph $G^{(2)}$

$G^{(k)}$(Figure 1.4).

Figure 1.5 shows an example of a perfect matching on $G^{(2)}$, where we have two kinds of edges. For $M \in \mathcal{M}(G)$, we say an edge $e \in M$ is an **impurity** if it is an edge between the vertices of $G_{1,T}$ and G_2. The number of impurities is constant on $\mathcal{M}(G^{(k)})$ and is equal to k.

We shall review the known results on the perfect matching problem on $G^{(k)}$. In [8], they consider two types of elementary moves which transforms a perfect matching on $G^{(k)}$ to a different one, and showed that any two perfect matchings are connected via a sequence of elementary moves. Then we can construct a Markov chain on $\mathcal{M}(G^{(k)})$ with the uniform stationary distribution. By MCMC simulation, we conjecture that all impurities tend to distribute near the terminals, more precisely, the number of perfect matchings is maximized when all the impurities are on the terminals. Ciucu [1, 2] studied the dimer-monomer problem mainly on the hexagonal lattice and

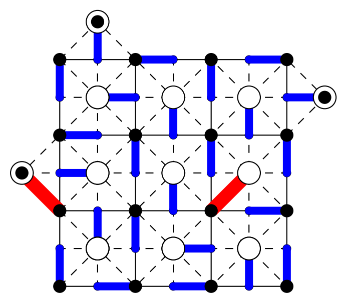

Figure 1.5: A Dimer Covering on $G^{(2)}$

found that monomers interact as if they were the charged particles in 2-dimensional electrostatics. Though his setting of the problem is different from ours, his works should have something to do with our conjecture. In [9], they studied the one impurity case, and derived a formula for computing the number of perfect matchings if the impurity is arbitrary fixed. From that formula, the conjecture above is generically correct for $k = 1$, in the sense that the probability of finding the impurity on the fixed site (under the uniform distribution on $\mathcal{M}(G^{(1)})$) decays exponentially away from the terminal.

In this paper, we study the $k \geq 2$ case and derive a formula to compute the number of perfect matchings if the impurities are fixed on the boundary(Theorem 1.2). Moreover, we study the behavior of the probability of finding the impurity when the size of the graph tends to infinity.

Notation

We collect the notations frequently used in this paper.

(1) $\triangle_G$ is the Laplacian on a weighted graph G :

$$(\triangle_G f)(v) = \sum_{w \in V(G)} c_{vw}(f(v) - f(w)) \tag{1.1}$$

where c_{vw} is the weight on the edge $(v, w) \in E(G)$ which we take $c_{vw} = 1$ in this paper.

K is the Laplacian on $G_{1,T}$ restricted on G_1 :

$$K := \triangle_{G_{1,T}}|_{G_1} = 1_{G_1} \triangle_{G_{1,T}} 1_{G_1}. \tag{1.2}$$

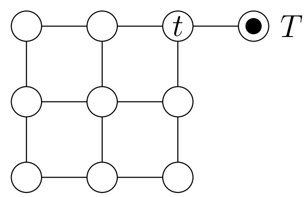

Figure 1.6: $G_{1,T}$ for one impurity case

1_S is the characteristic function on a set S.

$A(x, y)$ or $A_{x,y}$ is the matrix element of a matrix A.

(2) $I^{(1)}, I^{(2)}, \cdots, I^{(k)} \in E(G^{(k)})$ are the location of impurities, where $I^{(j)} = (I_1^{(j)}, I_2^{(j)})$, $I_i^{(j)} \in V(G_i)$, $j = 1, 2, \cdots, k$, $i = 1, 2$. For the one impurity case($k = 1$), we simply write $I = (I_1, I_2) \in E(G^{(1)})$, $I_i \in V(G_j)$, $i = 1, 2$.

$t_1, t_2, \cdots, t_{2k-1} \in V(G_1)$ are vertices of G_1 connected to the terminals $T_1, T_2, \cdots, T_{2k-1}$ respectively(Figure 1.2). For one impurity case, we simply write t(Figure 1.6).

In the following subsections, we summarize the results obtained in this paper.

1.2 One impurity case

We first recall the result in [9] where we set $k = 1$ and counted the number of matchings when the location of the impurity is given.

Theorem 1.1 *Let $x \in G_1$. Then the number $M(x)$ of perfect matchings on $G^{(1)}$ whose impurity satisfies $I_1 = x$ is equal to*

$$M(x) = |(K^{-1})(x, t) \det K|.$$

If we specify $I_1 = x \in G_1$, we have four possibilities of putting $I_2 \in G_2$, but the number of perfect matching is independent of the choice of that. In Section 3.1, we give an alternative proof of Theorem 1.1 by using the theory developped by Kenyon-Wilson [6, 7].

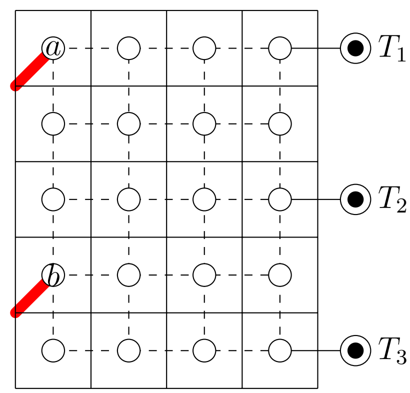

Figure 1.7: $G^{(2)}$ with fixed impurities

1.3 Two impurities case

Set $k = 2$ and let $a, b \in G_1$ be vertices on the boundary ∂G_1 of G_1 such that there are no terminals in between (Figure 1.7). We put the impurities such that their location $I^{(1)}, I^{(2)}$ satisfies $I_1^{(1)} = a$, $I_1^{(2)} = b$ and the other ends $I_2^{(1)}$, $I_2^{(2)}$ lie on the boundary ∂G_2 of G_2. We remark that, unlike the one impurity case, the number of perfect matchings becomes different when we put $I^{(1)}, I^{(2)} \notin \partial G_2$ (Figure 1.8). Let ∂C be the set of vertices on the boundary ∂G_1 of G_1 which lie between a and b (Figure 1.9).

Theorem 1.2 *Suppose the two impurities satisfy $I_1^{(1)} = a$, $I_1^{(2)} = b$ and $I_2^{(j)} \in \partial G_2$, $j = 1, 2$. Then the number $M(a, b)$ of the corresponding perfect matchings is given by*

$$M(a,b) = A(a,b) := \left| \det \begin{pmatrix} L_{a,t_1} & L_{a,t_2} & L_{a,t_3} \\ L_{\partial C,t_1} & L_{\partial C,t_2} & L_{\partial C,t_3} \\ L_{b,t_1} & L_{b,t_2} & L_{b,t_3} \end{pmatrix} \det(K) \right|$$

where

$$L_{x,t_i} = (K^{-1})(x, t_i), \quad x = a, b, \quad i = 1, 2, 3$$
$$L_{\partial C,t_i} = \sum_{y \in \partial C} (K^{-1})(y, t_i), \quad i = 1, 2, 3.$$

This determinantal expression implies that impurities are repulsive each other. We give two proofs of Theorem 1.2 : one is due to the hitting matrix

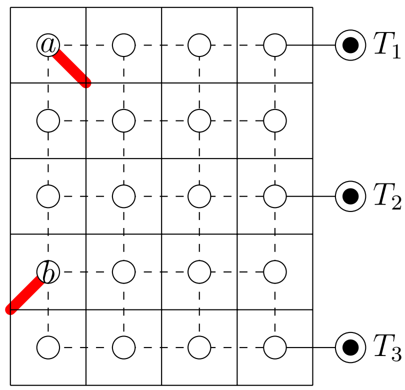

Figure 1.8: This impurity configuration is not allowed for Theorem 1.2.

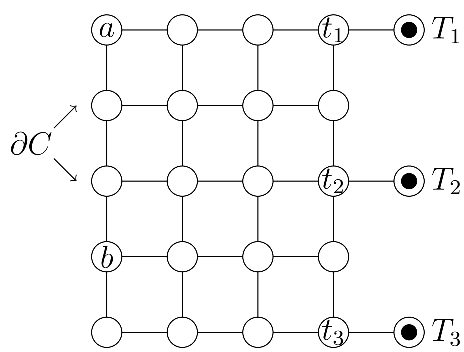

Figure 1.9: $G_{1,T}$ for two impurities

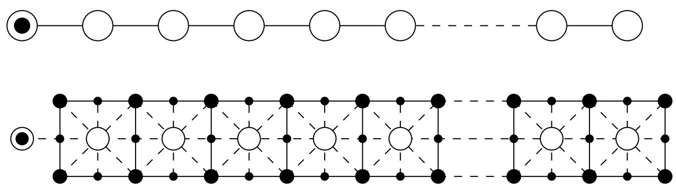

Figure 1.10: The case where G_1 is an one-dimensional chain and the corresponding $G^{(1)}$.

method by Fomin[4] in Section 2, and the other one by Kenyon-Wilson [6, 7] in Section 3.2. If $I_2^{(j)} \notin \partial G_2$ (as in Figure 1.8), then we have another formula given in Section 3.4. Essentially the same formula also holds for $k \geq 3$ case to be shown in Section 3.5. However, if some impurities are not on the boundary (i.e., $I_1^{(j)} \notin \partial G_1$ and $I_2^{(j)} \notin \partial G_2$), then our methods of proof do not apply and it would be difficult to compute the number of perfect matchings. On the other hand, if G_1 is the subgraph of the one-dimensional chain(Figure 1.10), we can compute the number of perfect matchings for arbitrary position of impurities(Section 3.6).

1.4 Large size limit

In the one impurity case, we consider the limit of $M(x)$ as the size of G_1 tends to infinity, for the following two graphs.

$$G_1 = \begin{cases} G_1^{sq}(n) := \{(x,y) \,|\, x = 1, 2, \cdots, n,\ y = 1, 2, \cdots, n\} & \text{(2-dim)} \\ G_1^{ch}(n) := \{1, 2, \cdots, n\} & \text{(1-dim)} \end{cases}$$

We can also study the case where G_1 is the $n \times m$-rectangle, provided n, m grow proportionally. We set the problem for the 2-dim case below, but the 1-dim case is formulated similarly. Connect the terminal T to $r = (1,1) \in G_1^{sq}(n)$ (for 1-dim, $r = 1 \in G_1^{ch}(n)$). We consider the following two problems.

(1) Compute $\lim_{n\to\infty} \mathbf{P}\left(I_1 = (x,y)\right)$ for fixed $(x,y) \in G_1^{sq}(n)$.

(2) We scale $G_1^{sq}(n)$ by $\frac{1}{n}$ so that it is contained by the unit square, and consider the probability of finding the impurity on a small region in it. In

other words, for any fixed $0 \leq c_1 < c_2 \leq 1$, $0 \leq d_1 < d_2 \leq 1$ we would like to compute

$$\mu([c_1, c_2] \times [d_1, d_2]) := \lim_{n \to \infty} \mathbf{P}\left(I_1 \in [c_1 n, c_2 n] \times [d_1 n, d_2 n]\right)$$

of a measure μ on $[0, 1]^2$.

For the first problem,

Theorem 1.3 *(1) (Example 3.6 in [9]) For the one-dimensional chain $G_1^{ch}(n)$, for fixed $j \in G_1^{ch}(n)$, we have*

$$\mathbf{P}\left(I_1 = j\right) = \frac{1}{4}\lambda_+^{-j}(1 + o(1)), \quad n \to \infty$$

where $\lambda_+ := 2 + \sqrt{3}$.
(2) For the two-dimensional grid $G_1^{sq}(n)$, for any fixed $(x, y) \in G_1^{sq}(n)$, we have

$$\lim_{n \to \infty} \mathbf{P}\left(I_1 = (x, y)\right) = 0.$$

Theorem 1.3 has the following implications. (i) for the one-dimensional chain, the probability of finding the impurity decays exponentially and it is localized near the terminal, (ii) for the two-dimensional grid, the probability spread if the size of the grid is large.
The perfect matching on $G^{(1)}$ is determined by a spanning tree T on a graph $G_{1,R}$ defined in Section 2.1(Theorem 2.1). And the difference between the one- and two-dimensional cases comes from that of the expectation value of the length l_T of T (Proposition 4.1). In one-dimensional case, it is bounded with respect to n, while it diverges in the logarithmic order in the two-dimensional case(Proposition 4.1(2)). This observation together with the Chebyshev's inequality also solves the second problem :

Theorem 1.4 *In both cases, μ is equal to the delta measure on the origin.*

In [3], they studied the lozenge tiling with a gap and showed that the correlation function behaves like $\frac{1}{r}$, where r is the distance between the gap and the boundary. In our case, the impurity tends to be attracted to the terminal T and not to the whole boundary, so that the situation is different. In the following sections, we prove those theorems mentioned above.

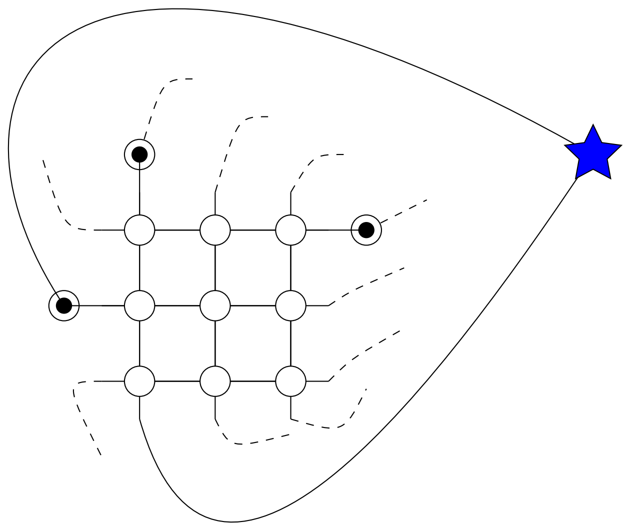

Figure 2.1: $G_{1,T,R}$: all vertices on $\partial G_{1,T}$ are connected to R.

2 Hitting Matrix

2.1 An extension of Temperley Bijection

We first recall the results in [10]. We consider an imaginary vertex R (called the root) in the outer face of $G_{1,T}$ and let $G_{1,T,R}$ be the graph obtained by connecting all vertices in $\partial G_{1,T}$ to the root R(Figure 2.1).

TI-tree is a tree on $G_{1,T,R}$ starting from the root R, which directly connects it to a terminal, and ends at a vertex of $G_{1,T}$. **TO-tree** is a tree on $G_{1,T,R}$ starting at a terminal and ends at the root R through a boundary vertex. **IO-tree** is a tree on $G_{1,T,R}$ starting at a vertex of G_1 and ends at the root R through a boundary vertex(Figure 2.2). Theorem 2.8 in [10], in

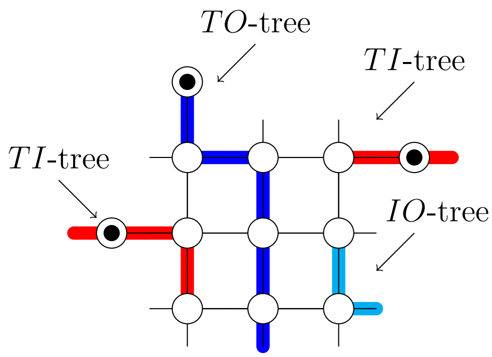

Figure 2.2: Example of TI-tree, TO-tree, and IO-tree. R is omitted.

a slightly different form, is :

Theorem 2.1 *We have a bijection between the following two sets.*

$$\begin{aligned}\mathcal{M}(G^{(k)}) &:= \{ \text{ perfect matchings on } G^{(k)}\} \\ \mathcal{F}(G^{(k)}, Q) &:= \{(T, S, \{e_j\}_{j=1}^k) \mid \ T \text{ : spanning tree on } G_{1,T,R}, \\ &\qquad S \text{ : spanning forest on } G_2, \\ &\qquad \{e_j\}_{j=1}^k \text{ : configuration of impurites, with condition } (\mathbf{Q}) \ \}\end{aligned}$$

Q *:*
(1) T is composed of k TI-trees, $(k-1)$ TO-trees, and the other ones are IO-trees,
(2) S is composed of k trees,
(3) T, S are disjoint of each other, and the k TI-trees of T and the k trees of S are paired by impurities.

Figure 2.3 is the spanning forest S on G_2 corresponding to the spanning tree of $G_{1,T,R}$ shown in Figure 2.2, and Figure 2.4 shows the corresponding perfect matching of $G^{(2)}$. Sometimes TI-tree and TO-tree sticks together to form another tree(Figure 2.5). For one impurity case($k = 1$), T is composed of a TI-tree and some IO-trees and the bijection in Theorem 2.1 is the same as Temperley's bijection(Figure 1.6).

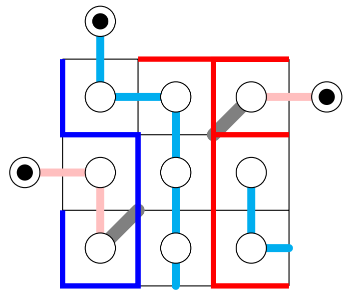

Figure 2.3: The spanning forest on G_2 corresponding to the spanning tree in Figure 2.2.

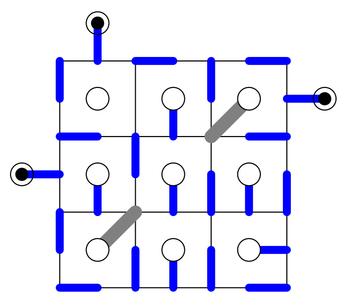

Figure 2.4: The corresponding perfect matching on $G^{(2)}$

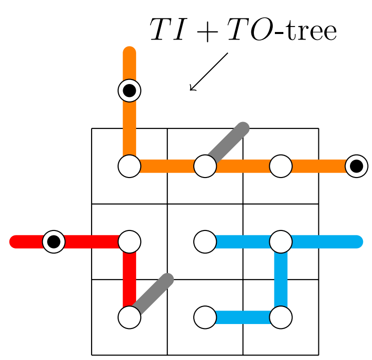

Figure 2.5: $TI + TO$-tree

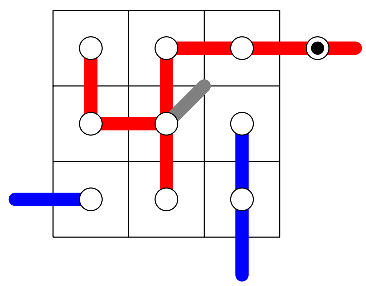

Figure 2.6: The case of one impurity

2.2 Proof of Theorem 1.2 using the hitting matrix method

By Theorem 2.1, $M(a,b)$ is equal to the number of elements in $\mathcal{F}(G^{(2)},Q)$ such that the corresponding spanning tree of $G_{1,T,R}$ is composed of (i) a TI-tree connecting a and T_1, (ii) a TI-tree connecting b and T_3, and (iii) a TO-tree connecting an element of ∂C and T_2 :

$$\begin{aligned} M(a,b) = \sharp\{F \,|\, F & \text{ is a spanning tree on } G_{1,T,R} \text{ composed of} \\ & \text{(i) TI-tree connecting } a \text{ and } T_1, \text{ (ii) TI-tree connecting } b \text{ and } T_3 \\ & \text{(iii) TO-tree connecting } \partial C \text{ and } T_2, \text{ and} \\ & \text{(iv) other IO-trees } \} \end{aligned} \tag{2.1}$$

Let $G_{1,R}$ be the dual graph of G_2 in the sense that we regard the outer face as a face of G_2 and put a dual vertex R there, which is called the root(Figure 2.7). $G_{1,R}$ is also obtained by identifying all terminals in $G_{1,T,R}$ with R.

Then the RHS of (2.1) is equal to the number of spanning trees T of $G_{1,R}$ with the following property : cutting all the edges connecting to R, yields a spanning forest T_1 of G_1 which satisfies (i) a and t_1 are connected, (ii) b and t_3 are connected, (iii) ∂C and t_2 are connected :

$$\begin{aligned} M(a,b) = \{T \,|\, T_1 & \text{ is a spanning forest on } G_1 \text{ such that} \\ & \text{(i) } a \text{ is connected to } t_1, \text{ (ii) } b \text{ is connected to } t_3, \\ & \text{(iii) } \partial C \text{ is connected to } t_2, \} \end{aligned}$$

By taking acount of Wilson's algorithm [11] generating the spanning trees

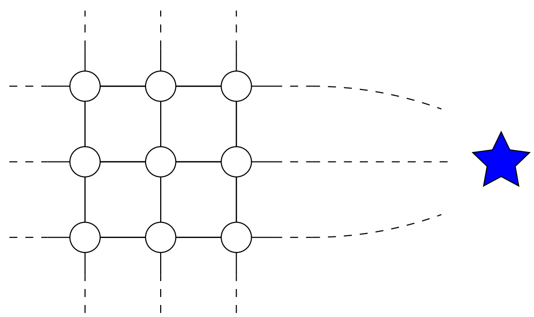

Figure 2.7: In $G_{1,R}$, all vertices on the boundary of G_1 are connected to R such that their degree = 4.

on $G_{1,R}$ uniformly at random, we have

$$\frac{M(a,b)}{\sharp\{\text{ spanning trees on } G_{1,R}\}} = \sum_{\sigma} (\text{ sgn } \sigma)\ w(\pi(a,t_{\sigma_1}))w(\pi(b,t_{\sigma_2}))w(\pi(\partial C,t_{\sigma_3}))1_{A(\sigma)}.$$

In the notation above, $\pi(x,y)$ is the walk from x to y, $|\pi|$ is the length of the walk π, $w(\pi) := 4^{-|\pi|}$ is the weight of the walk π, and $1_{A(\sigma)}$ is the indicator function of the condition $A(\sigma)$: " $\pi(b,t_{\sigma_2})$ is disjoint from the loop erased part of $\pi(a,t_{\sigma_1})$ and $\pi(c,t_{\sigma_3})$ is disjoint from the loop erased part of $\pi(b,t_{\sigma_2})$" :

$$A(\sigma) := \{\pi(b,t_{\sigma_2}) \cap LE(\pi(a,t_{\sigma_1})) = \emptyset,\ \pi(c,t_{\sigma_3}) \cap LE(\pi(b,t_{\sigma_2})) = \emptyset\}$$

Therefore by Theorem 7.2 in [4] and the matrix tree theorem : $\sharp\{\text{ spanning trees on } G_{1,R}\} = \det(K)$ we have

$$M(a,b) = \left| \det \begin{pmatrix} H_{a,t_1} & H_{a,t_2} & H_{a,t_3} \\ H_{b,t_1} & H_{b,t_2} & H_{b,t_3} \\ H_{\partial C,t_1} & H_{\partial C,t_2} & H_{\partial C,t_3} \end{pmatrix} \det(K) \right| \tag{2.2}$$

where H_{x,t_i} is the hitting probability from x to t_1 : probability of a simple random walk on $G_{1,R}$ starting at x hitting the root R through t_i, and

$$H_{\partial C,t_i} := \sum_{z \in \partial C} H_{z,t_i}.$$

To compute $H_{x,y}$, let Q be a matrix on G_1 given by

$$Q_{x,y} := \begin{cases} \frac{1}{4} & (x \sim y) \\ 0 & (\text{otherwise}) \end{cases}$$

Q is the transition probability matrix of the SRW on $G_{1,R}$ restricted to G_1. Since $4(I-Q) = K$, we have

$$\begin{aligned} H_{x,t_i} &= ((I-Q)^{-1})_{x,t_i} \cdot \frac{1}{4} \\ &= K^{-1}_{x,t_i} = L_{x,t_i} \end{aligned}$$

for $x \in G_1$, $i = 1, 2, 3$. Substituting it into (2.2) yields Theorem 1.2.

3 Alternative proofs by a Kenyon-Wilson's result

3.1 Kenyon-Wilson's results

We first recall the results by Kenyon-Wilson[6, 7]. Let G_C be a planar, finite, and weighted graph, and for $e = \{v, w\} \in E(G_C)$, let $w(e) = c_{vw}$ be the weight on that. We choose n vertices on the boundary of G_C, number those points anticlockwise, and let $N := \{1, 2, \cdots, n\}$ which is called the **node**. A graph with such a property is called a **circular planar graph**. Let $I := V(G_C) \setminus N$. We say a spanning forest g on G_C is a **grove** if each tree composing g contains at least an element of N. A grove g determines a planar partition (non-crossing partition) σ of N. We set

$$\begin{aligned} \mathcal{P} &:= \{ \text{ partitions of } \{1, 2, \cdots, n\}\} \\ \mathcal{P}_{pl} &:= \{ \text{ planar-partitions of } \{1, 2, \cdots, n\}\} \end{aligned}$$

For a grove g, let $w(g) := \prod_{e \in g} w(e)$ be its weight. For given planar partition $\sigma \in \mathcal{P}_{pl}$ of N, we consider all groves which realize the partition σ and let $Z_{G_C}(\sigma) := \sum_{g \in \sigma} w(g)$ be their weighted sum.

According to the decomposition $V(G_C) = N \cup I$, the Laplacian $\triangle_{G_C}$ on G_C can be written as

$$\triangle_{G_C} = \begin{pmatrix} F & G \\ H & K \end{pmatrix}. \tag{3.1}$$

We define the response matrix L by

$$L = GK^{-1}H - F.$$

L has the following physical meaning. We regard G_C as an electrical circuit with c_{vw} being the conductance on the edge $\{v, w\}$. For $i, j \in N$, set the voltage f on N by

$$f(x) = \begin{cases} 1 & (x = i) \\ 0 & (\text{otherwise}) \end{cases}, \quad x \in N.$$

Then $L_{i,j}$ is equal to the current flowing out of j.

$\sigma \in \mathcal{P}_{pl}$ is said to be **bipartite** if it satisfies the following condition : (i) it is possible to color the vertices in N contiguously into red and blue such that only the nodes of different colors are connected, (ii) if a node lies between red region and blue region, it can be split into two colors, and (iii) the number of each part in σ is at most two.

For bipartite σ, let $L_{X,Y}$ be the submatrix of L such that the color of the row(resp. column) is X (resp. Y). We then have the following theorem.

Theorem 3.1 (Theorem 3.1 in [7])
For bipartite $\sigma \in \mathcal{P}_{pl}$, we have

$$Z(\sigma) = |\det L_{R,B} \cdot \det(K)|.$$

Remark 3.2 *Theorem 3.1 in [7] treats more general case called the tripartite partition and write $Z(\sigma)$ in terms of Pfaffian. Theorem 3.1 also holds for non-planar graph G provided G satisfies the following condition :*

(A) *For any non-planar partition $\tau \in \mathcal{P} \setminus \mathcal{P}_{pl}$, we always have $Z_G(\tau) = 0$.*

3.2 One impurity case alternative proof of Theorem 1.1

For $x \in G_1$, we define a matrix K_x by

$$K_x(y, z) := \begin{cases} K(x, x) + 1 & ((y, z) = (x, x)) \\ K(y, z) & (\text{otherwise}) \end{cases}$$

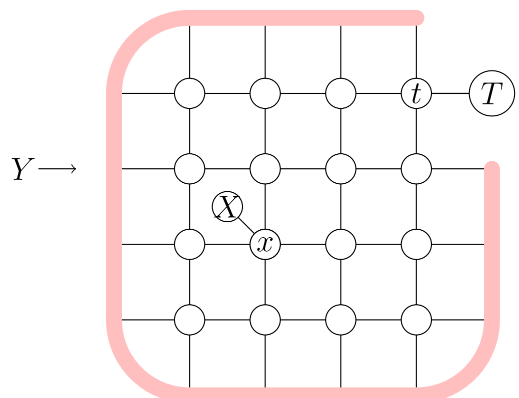

Figure 3.1: G_C in one impurity case : add X to x, identify all boundary vertices with Y

Lemma 3.3

$$M(x) = |(K_x^{-1})(x,t)\det(K_x)|.$$

Proof. We construct a circular planar graph G_C by the following procedure(Figure 3.1) : (i) add a new vertex X to x, (ii) add all boundary vertices to G_1, and identify them. Let Y be the corresponding vertex and we set

$$V(G_C) := I \cup N, \quad I = V(G_1), \quad N := \{T, X, Y\}.$$

Although G_C is not planar, the condition (A) is satisfied, for N contains only three elements so that we do not have non-planar partitions. Let $\sigma = XT|Y$ be a partition such that X and Y are connected and Y is isolated. By Theorem 2.1, we have $M(x) = Z_{G_C}(\sigma)$. Writing $\triangle_{G_C}$ as in (3.1), the response matrix L satisfies $L = GK_x^{-1}H - F$. Note that we added an extra vertex X to x, so that we have to use K_x in (3.1) instead of K. Since $F = 0$,

$$L_{X,T} = (GK_x^{-1}H)(X,T).$$

Using

$$(G)_{ij} = \begin{cases} -1 & (i \in N, j \in I, i \sim j) \\ 0 & (\text{otherwise}) \end{cases}$$

$$(H)_{ij} = \begin{cases} -1 & (i \in I, j \in N, i \sim j) \\ 0 & (\text{otherwise}) \end{cases}$$

and the fact t is the only vertex connected to the terminal T, we have

$$L_{X,T} = (K_x)^{-1}(x,t).$$

Substituting it to Theorem 3.1 yields the conclusion. □

Proof of Theorem 1.1 We show that the dependence of $L_{X,T} = (K_x)^{-1}(x,t)$ on x cancels with that of $\det K_x$. In fact, by taking the (x,x)-component of the resolvent equation

$$K_x^{-1} - K^{-1} = K_x^{-1}(K - K_x)K^{-1} \tag{3.2}$$

we have

$$(K_x^{-1})(x,x) - (K^{-1})(x,x) = -(K_x^{-1})(x,x)(K^{-1})(x,x)$$

thus

$$(K_x)^{-1}(x,x) = \frac{(K)^{-1}(x,x)}{1 + (K)^{-1}(x,x)}. \tag{3.3}$$

Moreover taking the (x,t)-component of (3.2) and using (3.3),

$$(K_x)^{-1}(x,t) = \frac{(K)^{-1}(x,t)}{1 + (K)^{-1}(x,x)}. \tag{3.4}$$

Let k_x be the submatrix of K_x by eliminating the x-th row and x-th column. By Cramer's formula,

$$(K)^{-1}(x,x) = \frac{\det(k^x)}{\det(K)}.$$

On the other hand, expanding the determinant yields

$$\det(K_x) = \det(K) + \det(k^x)$$

leading to

$$1 + (K)^{-1}(x,x) = \frac{\det(K) + \det(k^x)}{\det(K)} = \frac{\det(K_x)}{\det(K)}. \tag{3.5}$$

It then suffices to substitute (3.4), (3.5) to Lemma 3.3 to finish the proof of Theorem 1.1. □

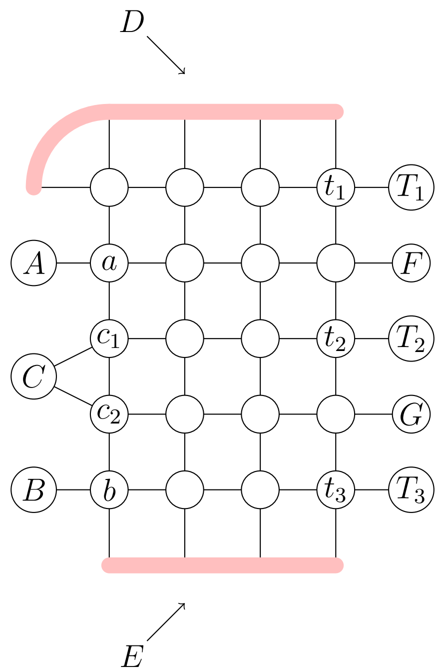

Figure 3.2: G_C in two impurity case

3.3 Two impurity case I alternative proof of Theorem 1.2

As we did in Section 3.3, we construct a circular planar graph G_C as follows(Figure 3.2) : (i) add all the boundary vertices to G_1, (ii) Let A, B be the boundary vertices attached to a and b respectively, and identify all boundary vertices lying between A and B, which we call C, and (iii) identify the other boundary vertices in their respective region. Let D, E, F, G be these vertices and set

$$V(G_C) := I \cup N, \quad I := V(G_1), \quad N := \{T_1, T_2, T_3, A, B, C, D, E, F, G\}.$$

Let $t_1, t_2, t_3, a, b \in I$ be the vertices connected to the nodes T_1, T_2, T_3, A, B respectively. By (2.1), $M(a, b)$ is equal to the number of the groves on G_C such that $A \leftrightarrow T_1, \partial C \leftrightarrow T_2, B \leftrightarrow T_3$ and D, E, F, G are isolated. The corresponding partition $\sigma \in \mathcal{P}_{pl}$ is bipartite (Figure 3.3) so that by Theorem

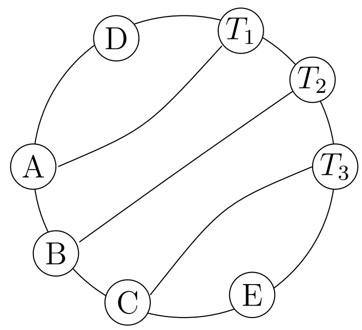

Figure 3.3: The corresponding planar partition σ. P, Q, R are red, A, B, C are blue, and D, E are split into red and blue.

3.1 we have

$$Z_{G_C}(\sigma) = \left| \det \begin{pmatrix} L_{A,T_1} & L_{A,T_2} & L_{A,T_3} \\ L_{C,T_1} & L_{C,T_2} & L_{C,T_3} \\ L_{B,T_1} & L_{B,T_2} & L_{B,T_3} \end{pmatrix} \cdot \det(K) \right|. \tag{3.6}$$

It suffices to compute these matrix elements of L. Under the same notation in (3.1), we have $L = -F + GK^{-1}H = GK^{-1}H$. Hence

$$L_{A,T_j} = (GK^{-1}H)_{A,T_j} = (K^{-1})_{a,t_j}, \quad j = 1, 2, 3 \tag{3.7}$$

likewise for L_{B,T_j}. Similarly

$$L_{C,T_i} = (GK^{-1}H)_{C,T_i} = \sum_{z \in \partial C} (K^{-1})_{z,t_i}. \tag{3.8}$$

By substituting (3.7), (3.8) into (3.6), we complete the proof of Theorem 1.2. $\square$

Remark 3.4 *If an impurity lies inside G_1, we generally have a non-planar partition τ with $Z(\tau) \neq 0$ so that the above argument is no longer valid.*

3.4 Two impurity case II : impurities lying "near" the boundary

In $G^{(2)}$, we consider the case in which both impurities satisfy $I_1^{(j)} \in \partial G_1$, $j = 1, 2$, but $I_2^{(1)} \notin \partial G_2$ (Figure 3.4). By Theorem 2.1, there are two types of

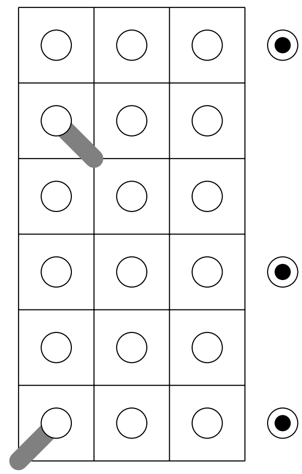

Figure 3.4: If $I_2^{(1)} \notin \partial G_2$, we have extra spanning trees.

corresponding spanning trees : one is given in Figure 3.5, whose contribution has been computed in Theorem 1.2, and the other one in Figure 3.6. As is explained in Figure 1.6, the tree connecting T_1 and T_2 in Figure 3.6 can be regarded as a composition of a TI-tree and a TO-tree. In this subsection we compute the contribution from the spanning trees in Figure 3.6. Let $c \in G_1$ be the vertex connected to $I_2^{(1)}$ through a diagonal edge(Figure 3.6).

Theorem 3.5 *Let $a, b \in \partial G_1$. The number $M(a,b)$ of perfect matchings satisfying $I_1^{(1)} = a$, $I_1^{(2)} = b$ and $I_2^{(1)} \notin \partial G_2$, $I_2^{(2)} \in \partial G_2$ is equal to*

$$\begin{aligned} M(a,b) &= A(a,b) + B(a,b) \\ B(a,b) &:= \left| \det \begin{pmatrix} L_{a,t_1} & L_{a,t_2} & L_{a,t_3} \\ L_{b,t_1} & L_{b,t_2} & L_{b,t_3} \\ L_{c,t_1} & L_{c,t_2} & L_{c,t_3} \end{pmatrix} \cdot \det(K) \right| \end{aligned}$$

where $A(a,b)$ is the one given in Theorem 1.2 and

$$L_{x,y} := (K^{-1})(x,y).$$

It is also possible to deal with the case where $I_2^{(j)} \notin \partial G_2$ $j = 1, 2$, with some extra terms computed similarly.

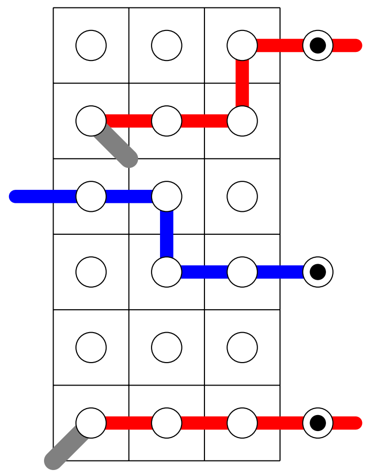

Figure 3.5: A spanning trees contributing to $A(a, b)$. IO-trees are omitted.

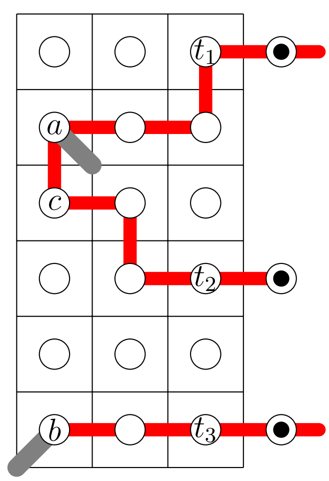

Figure 3.6: A spanning trees contributing to $B(a, b)$. IO-trees are omitted.

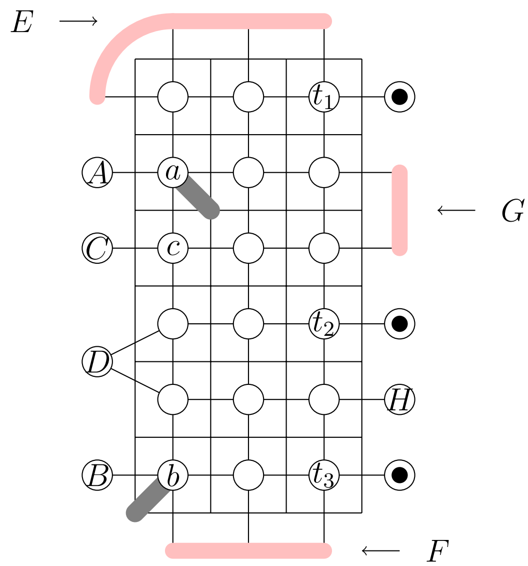

Figure 3.7: G_C to compute $B(a,b)$.

Proof. We compute $B(a,b)$ which is the number of the spanning trees on $G_{1,T,R}$ corresponding to the ones in Figure 3.6. As is done in subsection 3.3, we add all the boundary vertices to G_1, and let A, B, C be the boundary vertices connected to a, b, c respectively. We identify all boundary vertices between C and B, which we call D(Figure 3.7). Similarly, we define vertices E, F, G, H. Let

$$V(G_C) := I \cup N, \quad I = V(G_1), \quad N := \{T_1, T_2, T_3, A, B, C, D, E, F, G, H\}.$$

$B(a,b)$ is equal to the number of groves such that $T_1 \leftrightarrow A$, $T_2 \leftrightarrow C$, $T_3 \leftrightarrow B$, and the other nodes are isolated. Since the corresponding $\sigma \in \mathcal{P}_{pl}$ is bipartite, we can apply Theorem 3.1. □

3.5 More impurities case

It is possible to extend our argument in Section 3.3 to the case where the number of impurities satisfy $k \geq 3$, provided all impurities lie on the boundary. Let $(I_1^{(1)}, I_1^{(2)}, \cdots, I_1^{(k)}) = (a_1, a_2, \cdots, a_k)$, $a_j \in \partial G_1$, $j = 1, 2, \cdots, k$ be

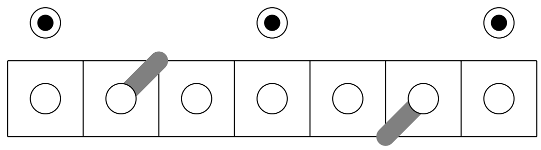

Figure 3.8: An example impurity configuration in one-dimensional chain

the position of impurities. Let ∂C_j, $j = 1, 2, \cdots, k-1$ be the set of vertices in ∂G_1 lying between a_j and a_{j+1}.

Theorem 3.6 *Let $k \geq 2$. If all impurities lie on the boundary and satisfy $I_1^{(j)} = a_j$, $a_j \in \partial G_1$, $I_2^{(j)} \in \partial G_2$, $j = 1, 2, \cdots, k$, then the number of corresponding perfect matchings is equal to*

$$M(a_1, a_2, \cdots, a_k) = \left| \det \begin{pmatrix} L_{a_1,t_1} & L_{a_1,t_2} & \cdots & L_{a_1,t_{2k-1}} \\ L_{a_2,t_1} & L_{a_2,t_2} & \cdots & L_{a_2,t_{2k-1}} \\ \cdots & & & \\ L_{a_k,t_1} & L_{a_k,t_2} & \cdots & L_{a_k,t_{2k-1}} \\ L_{\partial C_1,t_1} & L_{\partial C_1,t_2} & \cdots & L_{\partial C_1,t_{2k-1}} \\ \cdots & & & \\ L_{\partial C_{k-1},t_1} & L_{\partial C_{k-1},t_2} & \cdots & L_{\partial C_{k-1},t_{2k-1}} \end{pmatrix} \cdot \det(K) \right|.$$

We also have the analogue of Theorem 3.5.

3.6 One-dimension case

When G_1 is the one-dimensional chain, the impurities always lie on the boundary. Hence we can compute the number of perfect matchings for all configuration of impurities, by using the methods discussed in Sections 3.3, 3.4. For instance, we consider the case described in Figure 3.8. There are the two types spanning trees on $G_{1,T,R}$ given in Figures 3.9, 3.10. For each case we construct the circular planar graph G_C as we did in Sections 3.3, 3.4. The number of spanning trees in Figure 3.9 is equal to $A(a, b)$ given in Section 3.3. The number of spanning trees in Figure 3.10 is equal to $C(a, b)$, which is the number of groves satisfying $T_1 \leftrightarrow T_2$, $T_3 \leftrightarrow b$, and the other nodes are isolated.

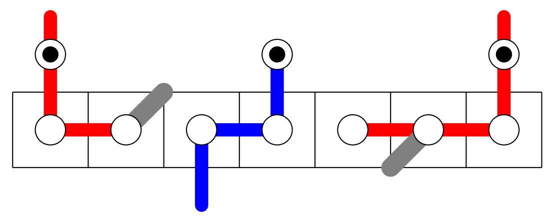

Figure 3.9: A spanning tree contributing to $A(a, b)$

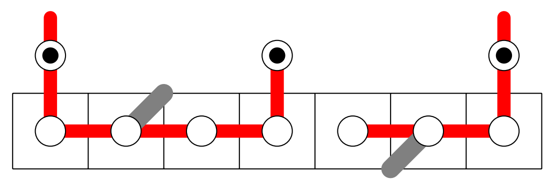

Figure 3.10: A spanning tree contributing to $C(a, b)$

4 Large scale limit

In this section we go back to the one impurity case and prove Theorems 1.3, 1.4. We take G_1 to be the $n \times m$ rectangle :

$$G_1 = G_1^{(n,m)} := \{(i, j) \,|\, 1 \le i \le n,\ 1 \le j \le m\}.$$

The terminal T is attached to the vertex $(1, 1)$. Let l_T be the length of the TI-tree of the spanning tree T given in Theorem 2.1, and let

$$A^{(n,m)} := (\triangle_{G_{1,T}}|_{G_1})^{-1}.$$

By Theorem 1.1,

$$(1)\ \mathbf{P}(I_1 = (x, y)) = \frac{A^{(n,m)}((x, y), (1, 1))}{4\sum_{x'=1}^{n}\sum_{y'=1}^{m} A_{n,m}((x', y'), (1, 1)) + 2} \tag{4.1}$$

$$(2)\ \mathbf{E}[l_T] = \sum_{x'=1}^{n}\sum_{y'=1}^{m} A^{(n,m)}((x', y'), (1, 1)). \tag{4.2}$$

We would like to study the behavior of $\mathbf{P}(I_1 = (x, y))$ and $\mathbf{E}[l_T]$ as $n = m$ tends to infinity.

Proposition 4.1 *Let $n = m$.*
(1) For fixed $(x, y) \in G_1^{(n,m)}$,

$$\lim_{n\to\infty} A^{(n,n)}((x,y),(1,1)) = A(x,y)$$
$$A(x,y) := \frac{2}{\pi^2}\int\int_{(0,\pi)^2} \frac{\sin(\theta x)\sin(\phi y)\sin\theta\sin\phi}{2-\cos\theta-\cos\phi} d\theta d\phi.$$

(2) Under the uniform distribution on the spanning trees,

$$\mathbf{E}[l_T] = \sum_{(x,y)\in G_1} A^{(n,n)}((x,y),(1,1)) = \frac{2}{\pi}\log n(1+o(1)).$$

as $n \to \infty$.

Proof. (1) The eigenvalues and the corresponding normalized eigenvectors of $A^{(n,m)}$ are given by

$$e_{kl} := 4 - 2\cos\left(\frac{k\pi}{n+1}\right) - 2\cos\left(\frac{l\pi}{m+1}\right),$$
$$\psi_{kl}(x,y) := \frac{2}{\sqrt{(n+1)(m+1)}}\sin\left(\frac{k\pi}{n+1}x\right)\sin\left(\frac{l\pi}{m+1}y\right),$$

$(k = 1, 2, \cdots, n,\ l = 1, 2, \cdots, m)$. Hence

$$\begin{aligned} & A^{(n,m)}((x,y),(1,1)) \\ = \quad & \frac{4}{(n+1)(m+1)}\sum_{k=1}^{n}\sum_{l=1}^{m}\frac{1}{e_{kl}} \times \\ & \times \sin\left(\frac{k\pi}{n+1}x\right)\sin\left(\frac{l\pi}{m+1}y\right)\sin\left(\frac{k\pi}{n+1}\right)\sin\left(\frac{l\pi}{m+1}\right) \\ \stackrel{n=m\to\infty}{\to} \quad & A(x,y). \end{aligned}$$

Here we note that the behavior of the integrand near the singularity $(\theta, \phi) = (0, 0)$ is

$$\frac{\sin(\theta x)\sin(\phi y)\sin\theta\sin\phi}{4-2\cos\theta-2\cos\phi} = \frac{\theta\cdot\phi}{\theta^2+\phi^2}\sin(\theta x)\cdot\sin(\phi y) + O(|\theta|+|\phi|)$$

so that it is removable.
(2) The sum of $A^{(n,m)}((x,y),(1,1))$ is equal to

$$\begin{aligned}&\sum_{x,y} A^{(n,m)}((x,y),(1,1))\\&= \frac{4}{(n+1)(m+1)} \sum_{k,l:\mathrm{odd}} \frac{1}{e_{kl}} \cdot \frac{\sin^2\left(\frac{k\pi}{n+1}\right)\sin^2\left(\frac{l\pi}{m+1}\right)}{\left(1-\cos\frac{k\pi}{n+1}\right)\left(1-\cos\frac{l\pi}{m+1}\right)}.\end{aligned}$$

In what follows, we let $m=n$. Due to the monotonicity of the integrand near the origin, we have, for a positive constant C,

$$\begin{aligned}&\sum_{x,y} A_{n,n}((x,y),(1,1))\\&\geq \frac{1}{\pi^2}\int_{\frac{1}{n}}^{\pi}\int_{\frac{1}{n}}^{\pi} \frac{\sin^2\theta\sin^2\phi}{(1-\cos\theta)(1-\cos\phi)} \cdot \frac{d\theta d\phi}{4-2\cos\theta-2\cos\phi} - C.\end{aligned}$$

Since the integrand behaves

$$\frac{\theta^2\phi^2}{\frac{\theta^2}{2}\cdot\frac{\phi^2}{2}} \cdot \frac{1}{\theta^2+\phi^2} = \frac{4}{\theta^2+\phi^2}$$

near the origin, we have as $n\to\infty$

$$\begin{aligned}&\frac{1}{\pi^2}\int_{\frac{1}{n}}^{\pi}\int_{\frac{1}{n}}^{\pi} \frac{\sin^2\theta\sin^2\phi}{(1-\cos\theta)(1-\cos\phi)} \cdot \frac{d\theta d\phi}{4-2\cos\theta-2\cos\phi}\\&= \frac{1}{\pi^2}\int_{\frac{1}{n}}^{\pi} dr\,\frac{4}{r}\int_0^{\frac{\pi}{2}} d\theta(1+o(1))\\&= \frac{2}{\pi}\log n(1+o(1)).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\sum_{xy} G(x,y) \leq \frac{1}{\pi^2}\int_{\frac{1}{n}}^{\pi}\int_{\frac{1}{n}}^{\pi} \frac{\sin^2\theta\sin^2\phi}{(1-\cos\theta)(1-\cos\phi)} \cdot \frac{d\theta d\phi}{4-2\cos\theta-2\cos\phi}\\&+\frac{4}{(n+1)^2}\sum_{l=1}^{\left[\frac{m+1}{2}\right]} \frac{\sin^2\frac{\pi}{n+1}\sin^2\frac{(2l-1)\pi}{m+1}}{\left(4-2\cos\frac{\pi}{n+1}-2\cos\frac{(2l-1)\pi}{m+1}\right)} \frac{1}{(1-\cos\frac{\pi}{n+1})(1-\cos\frac{(2l-1)\pi}{m+1})}\\&+(n\leftrightarrow m,\ k\leftrightarrow l)\\&=: I+II+III.\end{aligned}$$

The second term is estimated as

$$II = \frac{4}{(n+1)^2}\sum_l \frac{(1+\cos\frac{\pi}{n+1})(1+\cos\frac{(2l-1)\pi}{m+1})}{4-2\cos\frac{\pi}{n+1}-2\cos\frac{(2l-1)\pi}{m+1}}$$
$$\leq (Const.)\frac{4}{(n+1)^2}\sum_l \frac{(n+1)^2}{\pi^2+(2l-1)^2\pi^2} = O(1)$$

and similarly for III. Therefore the upper and lower bounds have the same leading order yielding

$$\mathbf{E}[l_T] = \sum_{x,y} A_{n,n}((x,y),(1,1)) = \frac{2}{\pi}\log n(1+o(1)).$$

$\square$

Proof of Theorem 1.3
(1) follows from [9], Example 3.6, and (2) follows from eq.(4.1) and Proposition 4.1(1), (2). $\square$

Proof of Theorem 1.4
Let $I_1 = \mathbf{r} \in G_1$ be the position of the impurity and take any $c > 0$. By Theorem 2.1 if $|\mathbf{r}| \geq cL$ then $l_T \geq cL$ so that by Chebychev's inequality

$$\mathbf{P}\left(|\mathbf{r}| \geq cn\right) \leq \mathbf{P}\left(l_T \geq cn\right) \leq \frac{1}{cn}\mathbf{E}[l_T].$$

For one-dimension, $\mathbf{E}[l_T] < \infty$ by Theorem 1.3(1) and for two-dimension $\mathbf{E}[l_T] = (const.)\log n(1+o(1))$, $n \to \infty$ by Proposition 4.1(2). Thus $\lim_{n\to\infty}\mathbf{P}(|\mathbf{r}| \geq cn) = 0$. $\square$

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